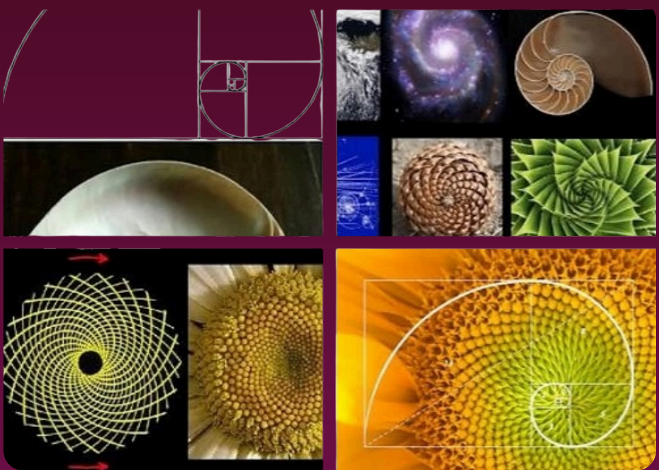
Fibonacci Numbers

The Fibonacci Numbers form a sequence that is one of the most famous and widely studied patterns in mathematics, appearing across many fields from number theory to nature and finance.

Definition and Sequence

The Fibonacci sequence is an infinite series of numbers where each number is the sum of the two preceding ones.



- **Initial Conditions (Starting Point):** The sequence conventionally begins with 0 and 1.

  $F_0=0$

  $F_1=1$

- **Recurrence Relation (The Rule):** For any number F_n in the sequence where $n \geqslant 2$, the rule is: $\mathbf{F_n=F_{n-1}+F_{n-2}}$
- **The Sequence:** 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377,...

History (The Origin)

The sequence is named after the Italian mathematician **Leonardo of Pisa** (c. 1170–1250), known as Fibonacci.

- **Indian Roots:** The sequence was described centuries earlier in **Indian mathematics** (as early as 200 BCE) by Sanskrit scholars like Pingala and later by Virahanka and Hemachandra, who used it in the analysis of rhythmic patterns (prosody) in Sanskrit poetry.

- **Western Introduction:** Fibonacci introduced the sequence to Western European mathematics in his 1202 book, Liber Abaci (The Book of Calculation). He presented it by solving a problem about the growth of an idealised rabbit population.

3 Key Mathematical Properties

A. Connection to the Golden Ratio (φ)

The most celebrated property of the Fibonacci sequence is its connection to the **Golden Ratio** (φ).

- **The Ratio Limit:** As you take the ratio of consecutive Fibonacci numbers, the result rapidly approaches the Golden Ratio: **$n \to \infty \lim F_{n-1} F_n = \varphi \approx 1.6180339887...$**

Binet's Formula (Closed-Form): The n-th Fibonacci number can be calculated directly using φ: **$F_n = 5\ \varphi^n - \varphi^n$**

where $\varphi = 21+5$ (the Golden Ratio) and $\varphi = 21-5$ (its conjugate).

B. Identities and Relations

Fibonacci numbers satisfy many identities, including **Cassini's Identity**:
$F_{n-1}F_{n+1} - F_n^2 = (-1)^n$

C. Relationship to Other Sequences

The Fibonacci numbers are closely related to the **Lucas Numbers** (L_n), which follow the same recurrence rule ($L_n = L_{n-1} + L_{n-2}$) but start with different initial values: $L_0 = 2$, $L_1 = 1$.

Applications (From Nature to Finance)

The Fibonacci sequence and the Golden Ratio appear in an astonishing range of fields.

Biology and Nature

Phyllotaxis: The spiral arrangement of leaves on a stem, seeds in a **sunflower head**, or florets in a cauliflower often contains a number of spirals that are consecutive Fibonacci numbers (e.g., 21 and 34, or 34 and 55).

Branching: The way trees branch or the structure of a drone bee's family tree follows the sequence.

Computer Science

Used in the analysis of the **Euclidean Algorithm** (where Fibonacci numbers are the worst-case input).

Basis for the **Fibonacci Search Technique**, an efficient method for finding the minimum of a unimodal function.

Used in data structures like the **Fibonacci Heap.**

Art and Architecture

The Golden Ratio is widely used in design (e.g., in the dimensions of the Parthenon or in Renaissance art) to create aesthetically pleasing proportions.

The **Fibonacci Spiral**, constructed by drawing quarter-circles inside squares whose side lengths are Fibonacci numbers, closely approximates the elegant **Golden Spiral** found in nature (like the shape of a nautilus shell).

Finance

Fibonacci Retracements: A tool in technical analysis used by traders to identify potential support and resistance levels in financial markets by using ratios derived from the sequence (e.g., 38.2%, 61.8%).